General analysis of self-dual solutions for the Einstein-Maxwell-Chern-Simons theory in $(1+2)$ dimensions

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The solutions of the Einstein-Maxwell-Chern-Simons theory are studied in (1+2) dimensions with the self-duality condition imposed on the Maxwell field. We give a closed form of the general solution which is determined by a single function having the physical meaning of the quasilocal angular momentum of the solution. This function completely determines the geometry of spacetime, also providing the direct computation of the conserved total mass and angular momentum of the configurations.

The (1+2)-dimensional general relativity has attracted considerable attention recently (see, e.g., [1] and references therein). This is explained by two main reasons. Firstly, since the discovery of the BTZ black hole solutions [2], the three-dimensional gravity became a helpful laboratory for the study of geometrical, statistical and thermodynamics properties of black holes. Secondly, the quantization of these models may give new insights into the general quantum gravity problem.

A number of generalizations of BTZ solution to the case of nontrivial electromagnetic field source were developed previously [4–7]. The aim of our present paper is to give a new general analysis of the self-dual Einstein-Maxwell solutions in three dimensions.

The Lagrangian 3-form,

$$L = \frac{1}{2}\,\mathcal{R}\,{*1} - \lambda\,{*1} - \frac{1}{2}F\wedge {*F} - \frac{\mu}{2}A\wedge F, \tag{1}$$

contains the Einstein-Hilbert term, the cosmological constant λ, and the standard Maxwell field $F = dA$ Lagrangian along with the Chern-Simons term with the coupling constant μ, [3]. Variation of L with respect to the coframe field ϑ^α and the electromagnetic potential A yields the system of field equations:

$$G_{\alpha\beta}\,{*\vartheta^\beta} + \lambda\,{*\vartheta_\alpha} = \Sigma_\alpha, \tag{2}$$

$$d\,{*F} + \mu\,F = 0. \tag{3}$$

Here $\Sigma_\alpha = \frac{1}{2}\left[(e_\alpha\rfloor F)\wedge {*F} - F\wedge e_\alpha\rfloor {*F}\right]$ is the Maxwell field energy-momentum 2-form, and $G_{\alpha\beta}$ is the Einstein tensor.

In the study of the "spherically"-symmetric solutions, we choose the local coordinates (t, r, ϕ) and make the general ansatz for the coframe 1-form,

$$\vartheta^0 = f\,dt, \quad \vartheta^1 = g\,dr, \quad \vartheta^2 = h\,(d\phi + a\,dt), \tag{4}$$

and for the Maxwell field

$$F = E\,\vartheta^0\wedge\vartheta^1 + B\,\vartheta^1\wedge\vartheta^2, \tag{5}$$

Here f, g, h, a and E, B are the functions of the radial coordinate r.

Without any loss of generality it will be convenient to absorb the metric function $g(r)$ by the simple redefinition of the radial coordinate:

$$\rho = \int g(r)\,dr \qquad (\text{hence } \vartheta^1 = d\rho). \tag{6}$$

¿From now on, the derivatives w.r.t. new coordinate ρ will be denoted by prime.

After all these preliminaries, the Einstein field equations read explicitly

$$-\,\frac{1}{2}\,\beta' - \beta\gamma = -EB, \tag{7}$$

$$\gamma' + \gamma^2 + \frac{1}{4}\,\beta^2 + \lambda = -\,\frac{1}{2}\,(E^2 + B^2), \tag{8}$$

$$\alpha' + \alpha^2 - \frac{3}{4}\,\beta^2 + \lambda = \frac{1}{2}\,(E^2 + B^2), \tag{9}$$

$$-\,\alpha\gamma - \frac{1}{4}\,\beta^2 - \lambda = \frac{1}{2}\,(E^2 - B^2), \tag{10}$$

and this system is supplemented by the (modified) Maxwell equations:

$$-\,B' - \alpha B + \beta E + \mu E = 0, \tag{11}$$

$$-\,E' - \gamma E + \mu B = 0. \tag{12}$$

Here we introduced the functions

$$\alpha = \frac{f'}{f}, \quad \beta = \frac{a' h}{f}, \quad \gamma = \frac{h'}{h}. \tag{13}$$

which actually describe the Levi-Civita connection coefficients. The remarkable feature is that the complete Einstein-Maxwell system (7)-(12) involves no metric functions (i.e., f, g, h, a), but only the connection combinations α, β, γ.

Let us assume "self-duality" of the electromagnetic field:

$$E = k\,B, \qquad \text{with} \qquad k^2 = 1. \tag{14}$$

Substituting this into (11)-(12) and (10), we find that the two unknown functions are expressed in terms of the third:

$$\alpha = \frac{k}{2}\,\beta + \ell, \quad \gamma = -\,\frac{k}{2}\,\beta + \ell. \tag{15}$$

Here we denote $\ell := \pm\sqrt{-\lambda}$.

Taking into account the algebraic relations (14) and (15), we are left with two essential equations for determining the functions β and B. Explicitly, the equations (7) and (12) are reduced to

$$\beta' - k\,\beta^2 + 2\ell\,\beta = 2k\,B^2, \tag{16}$$
$$(B^2)' - k\,\beta\,B^2 + 2\ell\,B^2 = 2k\mu\,B^2. \tag{17}$$

This system of nonlinear coupled equations is simplified with the help of the substitution

$$\beta = \frac{1}{\omega}, \quad B^2 = \frac{k}{2\omega}\,\frac{\varphi'}{\varphi}, \tag{18}$$

which yields for the new functions φ and ω the linear equations:

$$\varphi'' = 2k\mu\,\varphi'. \tag{19}$$
$$\omega' + \left(\frac{\varphi'}{\varphi} - 2\ell\right)\omega + k = 0. \tag{20}$$

Multiplying (20) by $\varphi\,e^{-2\ell\rho}$, we easily obtain the general solution

$$\omega = \frac{k\Omega}{\varphi\,e^{-2\ell\rho}}, \quad \text{with} \quad \Omega := c_0 - \int^{\rho} d\tilde\rho\,\varphi(\tilde\rho)\,e^{-2\ell\tilde\rho}. \tag{21}$$

Note that in fact it is not necessary to know the explicit form of φ when solving (20). At the same time, of course, the equation (19) is straightforwardly integrated. Depending on μ, it admits two solutions:

$$\varphi = \rho + \rho_0, \quad \text{when} \quad \mu = 0, \tag{22}$$
$$\varphi = 1 + u_0 e^{2k\mu\rho}, \quad \text{when} \quad \mu \neq 0. \tag{23}$$

Here c_0, ρ_0, u_0 are integration constants. It is worthwhile to note that an overall constant factor is irrelevant for φ because this function appears everywhere only through the ratio (18).

Quite remarkably, however, we will not need the explicit form of φ till the very end of our analysis. Such a formulation is extremely convenient since it makes it possible to treat the cases of standard Maxwell theory with $\mu = 0$, and the Maxwell-Chern-Simons with $\mu \neq 0$ simultaneously.

It remains to integrate the equations for the metric functions (13). This is straightforward, and using (21) in (15), we find:

$$f = f_0\,e^{\ell\rho}\,\Omega^{-\frac{1}{2}}, \tag{24}$$
$$h = h_0\,e^{\ell\rho}\,\Omega^{\frac{1}{2}}, \tag{25}$$
$$a = \frac{k\,f_0}{h_0}\,\Omega^{-1} - a_0. \tag{26}$$

For completeness, the magnetic field reads:

$$B^2 = \frac{\varphi'}{2}\,e^{-2\ell\rho}\,\Omega^{-1}. \tag{27}$$

Here f_0, h_0, a_0 are integration constants.

The first main result which we learned in our study, is that a general "spherically"-symmetric (rotating, for nontrivial a) solution

$$ds^2 = -\left(\vartheta^0\right)^2 + \left(\vartheta^1\right)^2 + \left(\vartheta^2\right)^2 \tag{28}$$

of the Einstein-Maxwell (with or without Chern-Simons term) field equations is always represented solely in terms the function φ.

Because of such an important role played by φ, it would be interesting to find out its physical meaning. The latter is revealed in the analysis of the quasilocal mass and angular momentum which characterize our general solution.

We refer the reader to [8] for a comprehensive discussion of the conserved quantities for gravitating systems within the framework of Hamiltonian formulation of general relativity theory. As a first step, let us use the coordinate freedom and replace ρ by a new radial coordinate defined by

$$r = h(\rho). \tag{29}$$

Then a nontrivial metric function g will reappear in the coframe (4) [and hence in the metric (28)]. Using (25) we find explicitly

$$g = \frac{d\rho}{dr} = \left(\ell\,r - \frac{h_0^2}{2r}\,\varphi\right)^{-1}. \tag{30}$$

Now we can write the quasilocal angular momentum at a distance r, which reads

$$j(r) = \frac{g^{-1}\,r^3}{f}\,\frac{da}{dr}, \tag{31}$$

in our notations. Using (24)-(26) and (29)-(30), we find

$$j(r) = k\,h_0^2\,\varphi. \tag{32}$$

Clearly, one should invert (29) and use $\rho = \rho(r)$ in (32), or alternatively, one can consider the angular momentum j as a function of ρ.

The quasilocal energy is given, in our notations, by the difference

$$E(r) = g_0^{-1} - g^{-1}, \tag{33}$$

where the first term describes the contribution of the background "empty" spacetime. The latter, as usually, is given by $g_0^{-1} = \ell\,r$. Making use of (30), we obtain explicitly

$$E(r) = \frac{h_0^2}{2r}\,\varphi = \frac{k}{2r}\,j(r). \tag{34}$$

Finally, the quasilocal mass is determined by the expression

$$m(r) = 2\,f\,E(r) - j\,a. \tag{35}$$

Substituting (24), (26), (32) and (34), we arrive at the result:

$$m(r) = a_0\,j(r). \tag{36}$$

We thus have demonstrated that the function φ, which determines the spacetime geometry via (21) and (24)-(26), is also determining *all* the quasilocal quantities of

the gravitating system: its energy, mass and angular momentum. They turn out to be proportional to each other, describing a sort of extremal configuration. Because of the relation (32), one can say that the angular momentum $j(r)$ underlies the construction of self-dual Einstein-Maxwell equations: given this function, the metric and electromagnetic field are described by (24)-(27) with $j(r)$ inserted.

The total angular momentum and mass are defined by the limits $J := j|_{r\to\infty}$ and $M := m|_{r\to\infty}$, respectively. In order to find these quantities, one does not need to obtain the explicit exact form of the inverse coordinate transformation $\rho(r)$ from (29). It is sufficient to investigate the approximate behaviour of $\varphi(r)$ and $\Omega(r)$ for large values of r, which is always clear directly from the inspection of (19)-(21).

In particular, one can immediately verify that the limiting value $\Omega|_{r\to\infty}$ is equal either infinity or c_0, depending on the values of μ and ℓ. Consequently, the integration constant a_0 should be equal either 0, or $\frac{kf_0}{h_0c_0}$, providing the required asymptotic vanishing of the metric function $a(r)$. Correspondingly, one finds that the quasilocal mass m vanishes for many configurations.

The quasilocal angular momentum $j(r)$ (or the function $\varphi(r)$) diverges, in general, for $r \to \infty$. However, the direct analysis of (19)-(21) shows that J is finite for all the solutions with $k\mu < 0$. Actually, there are two large classes of such configurations: (A) $k\mu < 0, \ell = 0$, then $J = kh_0^2$ and $M = 0$, and (B) $k\mu < 0, \ell > 0$, then $J = kh_0^2$ and $M = \frac{f_0h_0}{c_0}$. Imposing the standard asymptotic condition $f^2g^2|_{r\to\infty} = 1$, one finds $a_0 = k\ell$, and thus the solutions of the class (B) are all characterized (irrespectively of the value of the Chern-Simons coupling constant μ) by $M^2 = \ell^2 J^2$. This class also contains the extremal BTZ solution, as a particular case when the electromagnetic field is absent. [The general non-extremal BTZ solution cannot be recovered because of the algebraic relations (15) which necessarily hold for the self-dual electromagnetic field].

Summarizing, we have obtained a general solution of the Einstein-Maxwell-Chern-Simons theory in $(1+2)$ dimensions which covers all the particular cases studied previously. The form of the solution (24)-(27), (21) is transparent and easy to analyse: everything is determined by a single function $\varphi(\rho)$ which has a clear physical meaning as the quasilocal angular momentum of the gravitational field configuration. The computation of the total mass and angular momentum is straightforward and it involves only the analysis of the asymptotic behaviour of φ.

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